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r-ENTROPY, EQUIPARTITION, AND ORNSTEIN'S ISOMORPHISM THEOREM IN $\mathbf{R}^n$

BY

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ABSTRACT

A new approach is given to the entropy of a probability-preserving group action (in the context of $\mathbf{Z}$ and of $\mathbf{R}^n$), by defining an approximate "r-entropy", $0 < r < 1$, and letting $r \to 0$. If the usual entropy may be described as the growth rate of the number of essential names, then the r-entropy is the growth rate of the number of essential "groups of names" of width $\leqq r$, in an appropriate sense. The approach is especially useful for actions of continuous groups. We apply these techniques to state and prove a "second order" equipartition theorem for $\mathbf{Z}^m \times \mathbf{R}^n$ and to give a "natural" proof of Ornstein's isomorphism theorem for Bernoulli actions of $\mathbf{Z}^m \times \mathbf{R}^n$, as well as a characterization of such actions which seems to be the appropriate generalization of "finitely determined".

1. Introduction

This work arose in an attempt to answer several vague questions.

(A) The entropy of an ergodic, probability-preserving flow ϕ is defined by the formula $h_0(\phi) = h_0(\phi_1)$, that is, the entropy of its time 1 transformation. Thus, the connection of entropy with the information in a continuous orbit is not apparent. Is there any definition which exhibits this?

(B) In a similar vein, is there a good analog for flows of the Equipartition Theorem of Shannon and Macmillan?

(C) Is there a proof of Ornstein's Isomorphism Theorem for Bernoulli flows which doesn't use the theorem for transformations, but works directly with flows?

Answers presumably would open the possibility of generalizing much of the ergodic theory now known (for transformation, flows, and actions of discrete groups) to actions of continuous groups.

It turns out that these questions have what may be regarded as satisfactory affirmative answers. In the present paper these answers are given, in the context

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of $\mathbf{R}^n$. In future papers it will be shown how the techniques developed here may be extended to give similar results for a large class of (locally compact) groups. In view of recent work of Ornstein and Weiss [7], the proper setting is probably unimodular amenable groups.

Descriptions of earlier stages of this work were given in [1] and [2]. The author is grateful to those who suffered through various preliminary versions during the past year. Special thanks are due to D. Ornstein who, in addition to general discussion and encouragement, made two specific important contributions: Proposition 2.3, and also the present version of the definition of "semifinitely determined", which replaced my somewhat less natural version; and to my students M. Gerber and J. Stroik, who read the manuscript critically and suggested numerous improvements.

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2. r-entropy and r-equipartition in Z

Let T be an ergodic m.p.t. on (X, μ), and $\mathcal{P}$ a partition (all partitions will be measurable and finite). The usual definition of the entropy of the process $(T, \mathcal{P})$ is given as

$$h_0(T, \mathcal{P}) = \limsup_{N \to \infty} \frac{1}{N} H\left(\bigvee_{j=1}^{N} T^{-j}\mathcal{P} \right),$$

where $H(\mathcal{Q})$ is defined for any family $\mathcal{Q}$ of disjoint measurable sets by $H(\mathcal{Q}) = -\Sigma_{Q \in \mathcal{Q}} \mu(Q) \log \mu(Q)$.

Now choose some $r > 0$, and consider a collection $\mathcal{B}$ of disjoint $\bigvee_{j=1}^{N} T^{-j}\mathcal{P}$-measurable sets each having diameter $\leqq r$ with respect to the normalized Hamming metric on the $\mathcal{P}$-N-names of points: that is,

$$x, y \in B \in \mathcal{B} \Rightarrow d_N^{\mathcal{P}}(z, y) = \frac{1}{N} |\{j : 1 \leqq j \leqq N \text{ and } \mathcal{P}(T^j x) \neq \mathcal{P}(T^j y)\}| \leqq r.$$

Such a family we call a $(\mathcal{P}, N, r)$ family. Suppose also we ask that $\mu(\bigcup \mathcal{B}) > 1 - \varepsilon$, for small ε. How small can $(1/N)H(\mathcal{B})$ get?

(It should be noted that if instead of $H(\mathcal{B})$ we took $H(\mathcal{B} \cup \{X \sim \bigcup \mathcal{B}\})$, or calculated $H(\mathcal{B})$ with respect to conditional measure on $\bigcup \mathcal{B}$, the difference would go to zero with ε, so there would be no effect on what follows.)

2.1. Definition. We define $h_r(T, \mathcal{P})$, the "r-entropy," as the infimum of numbers b such that for every $\varepsilon > 0$, $\exists N_0$ such that if $N > N_0$ there exists a $(\mathcal{P}, N, r)$ family $\mathcal{B}$ with $\mu(\bigcup \mathcal{B}) > 1 - \varepsilon$ and $(1/N)H(\mathcal{B}) \leqq b$: or, more succinctly,

$$h_r(T, \mathcal{P}) = \sup_{\varepsilon > 0} \overline{\lim_{N \to \infty}} \inf \left\{ \frac{H(\mathcal{B})}{N} : \mathcal{B} \text{ a } (\mathcal{P}, N, r) \text{ family with } \mu(\mathcal{B}) > 1 - \varepsilon \right\}.$$

Clearly, $h_r(T, \mathcal{P}) \leqq h_0(T, \mathcal{P})$.

Alternately, taking a hint from the Equipartition Theorem, we make another definition.

2.2. Definition. For $r > 0$ define $k_r(T, \mathcal{P})$, the "r-count," as the same supremum, but with $\log |\mathcal{B}|$ instead of $H(\mathcal{B})$. From Macmillan's theorem, $k_r(T, \mathcal{P}) \leqq h_0(T, \mathcal{P})$, and also clearly $h_r(T, \mathcal{P}) \leqq k_r(T, \mathcal{P})$. Furthermore, the number of atoms of $\bigvee_{j=1}^{N} T^{-j}\mathcal{P}$ in a single $B \in \mathcal{B}$ is dominated by $\binom{N}{[Nr]} |\mathcal{P}|^{[Nr]}$, by counting the ways of distributing the disagreements. Here $\binom{N}{M}$ is a binomial coefficient, and $[\cdot]$ means "greatest integer $\leqq \cdot$".

Lemma. $\lim_{r \to 0} h_r(T, \mathcal{P}) = \lim_{r \to 0} k_r(T, \mathcal{P}) = h_0(T, \mathcal{P})$.

Proof. If $\mathcal{B}$ is a $(\mathcal{P}, N, r)$ family, then each $B \in \mathcal{B}$ contains no more than $\binom{N}{[rN]} |\mathcal{P}|^{[Nr]}$ different atoms. Suppose $\bigcup \mathcal{B}$ has measure $> 1 - \varepsilon/2$. For large N, Macmillan's theorem gives a $\bigvee_{j=1}^{n} T^{-j}\mathcal{P}$-measurable set E of measure $> 1 - \varepsilon/2$ so that all atoms in E have measure $< 2^{-(h(T,\mathcal{P}) - \varepsilon/2)N}$. Consider the $(\mathcal{P}, N, r)$ family $\mathcal{C} = \{B \cap E : B \in \mathcal{B}\}$. Then

$$\begin{aligned} H(\mathcal{C}) &= -\sum_B (\log \mu(B \cap E))\mu(B \cap E) \\ &= H(\mathcal{B}) - \sum_B \left(\log \frac{\mu(B \cap E)}{\mu(B)}\right) \mu(B) + \sum (\log \mu(B \cap E))\mu(B \cap (X \sim E)). \end{aligned}$$

The total measure of those B whose intersection with E is of proportion $1 - \sqrt{\varepsilon}$ must be of measure $< \sqrt{\varepsilon}$. So

$$\frac{H(\mathcal{C})}{N} \leqq \frac{H(\mathcal{B})}{N} - \frac{(\log(1 - \sqrt{\varepsilon}))(1 - \sqrt{\varepsilon})}{1} + \log |\mathcal{P}| \varepsilon. \qquad \square$$

Some further properties of these functions, easy to check, are:

(a) $h_r(T, \mathcal{P}) = k_r(T, \mathcal{P}) = 0$ if $r \geqq 1$,

(b) $h_r(T, \mathcal{P})$ and $k_r(T, \mathcal{P})$ are monotone nonincreasing functions of r,

(c) if $\mathcal{P}$ refines $\mathcal{Q}$ then $h_r(T, \mathcal{P}) \geqq h_r(T, \mathcal{Q})$, and for any fixed $r < 1$ we have $\sup_{\mathcal{P}} h_r(T, \mathcal{P}) = h_0(T)$, and similarly for $k_r(T, \mathcal{P})$. This involves a little argument, but since we won't use this result, we won't make that argument here.

It is important to note that, unlike $h_0(T, \mathcal{P})$, we do *not* have

$$h_r\left(T, \bigvee_{j=1}^{N} T^{-j}\mathcal{P}\right) = h_r(T, \mathcal{P}) \quad \text{or} \quad k_r\left(T, \bigvee_{j=1}^{N} T^{-j}\mathcal{P}\right) = k_r(T, \mathcal{P}).$$

2.3. Proposition. (D. Ornstein) *$k_r(T, \mathcal{P})$ is convex (and hence: continuous, and strictly monotonic) until it hits* 0.

Proof. Choose a $(\mathcal{P}, N, r)$ family $\mathcal{A}$ covering X up to δ, and a $(\mathcal{P}, N, s)$ family $\mathcal{B}$ covering X up to δ. Let $M = 2LN$, where N is large. Let $\tilde{\mathcal{A}} = \mathcal{A} \cup \{X \sim \bigcup \mathcal{A}\}$, $\tilde{\mathcal{B}} = \mathcal{B} \cup \{X \sim \bigcup \mathcal{B}\}$. Then choose L so large that, except for a set S of measure $< \delta$, the number of the first L blocks of $\mathcal{P}$ names of length N which fall in $X \sim \bigcup \mathcal{A}$ is less than a fraction 2δ of those blocks, and similarly for the second L blocks of length N which fall in $X \sim \bigcup \mathcal{B}$. Then if we define a family of sets $\mathcal{C}$ be selecting some member of $\tilde{\mathcal{A}}$ for each of the first L blocks of length N and some number of $\tilde{\mathcal{B}}$ for each of the second L blocks of length N, but excluding the set S, then $\mathcal{C}$ is a $(\mathcal{P}, M, (r+s)/2 + 2\delta)$ family, and

$$\frac{\log|\mathcal{C}|}{M} \leqq \frac{1}{2}\left(\frac{\log|\mathcal{A}|}{N} + \frac{\log|\mathcal{B}|}{N}\right).$$

It follows that

$$\lim_{\delta \to 0} k_{(r+s)/2+\delta}(T, \mathcal{P}) \leqq \frac{1}{2}(k_r(T, \mathcal{P}) + k_s(T, \mathcal{P})).$$

From this one easily sees that $k_r(T, \mathcal{P})$ is a convex function. □

We wish to prove an "equipartition theorem" for r-entropy: first a weaker form, which will then be improved.

2.4. Proposition. *Let T be ergodic on (X, μ). Let $\mathcal{P}$ be a partition, and $r > 0$. Then for any $\varepsilon > 0$ there exist N_0 and $\delta > 0$ such that if $N > N_0$ and $\mathcal{B}$ is a $(\mathcal{P}, N, r)$ family of measure $> 1 - \delta$ satisfying $\log|\mathcal{B}|/N < k_r(T, \mathcal{P}) + \delta$, then the B in $\mathcal{B}$ such that*

$$k_r(T, \mathcal{P}) - \varepsilon < -\frac{\log \mu(B)}{N} < k_r(T, \mathcal{P}) + \varepsilon$$

have total measure $\geqq 1 - \varepsilon$.

Remark. In view of the usual equipartition theorem, the conclusion may be rephrased "those $B \in \mathcal{B}$ which contain between $2^{N(h_0(T,\mathcal{P}) - k_r(T,\mathcal{P}) - \varepsilon)}$ and $2^{N(h_0(T,\mathcal{P}) - k_r(T,\mathcal{P}) + \varepsilon)}$ atoms of $\mathcal{P}_1^N$ form a set of measure $> 1 - \varepsilon$."

PROOF OF PROPOSITION 2.4. Write b for $k_r(T, \mathcal{P})$. Fix $\alpha > 0$. Let N and δ be chosen, and let $\mathcal{B}$ be a $(\mathcal{P}, N, r)$ family of measure $> 1-\delta$ with fewer than $2^{N(b+\delta)}$ members. Let $\mathcal{B}^+$ be those $B \in \mathcal{B}$ for which $\mu(B) \geqq 2^{-N(b-\alpha)}$. We show that if N is sufficiently large and δ sufficiently small then $\mu(\bigcup \mathcal{B}^+) \leqq \alpha$.

Suppose not. Fix N and δ for the moment. Choose L so large that for all x in a set X_0 of measure $> 1-\delta$ we have

$$(1) \qquad |\{j : 1 \leqq j \leqq NL,\ T^j x \in \textstyle\bigcup \mathcal{B}^+\}| > \alpha NL,$$

$$(2) \qquad |\{j : 1 \leqq j \leqq NL,\ T^j x \in X \sim \textstyle\bigcup \mathcal{B}\}| > \delta NL.$$

For $x \in X_0$, we concatenate names of length L with different starting points. Then (1) and (2) imply

$$(1)' \qquad \left|\left\{k : 0 \leqq k \leqq N-1, \text{ and } T_x^{k+jN} \in \textstyle\bigcup \mathcal{B}^+ \text{ for at least } \frac{\alpha}{2} L \text{ values of } j \text{ between 1 and } L\right\}\right| \text{ is greater than } \frac{\alpha}{4} N,$$

$$(2)' \qquad \left|\left\{k : 0 \leqq k \leqq N-1, \text{ and } T_x^{k+jN} \in X \sim \textstyle\bigcup \mathcal{B} \text{ for less than } \frac{4\delta}{\alpha} L \text{ values of } j \text{ between 1 and } L\right\}\right| \text{ is greater than } \frac{\alpha}{4}.$$

Combining, we get some $k(x)$, $0 \leqq k \leqq N-1$, such that $T^{k(x)+jN}$ is in $\bigcup \mathcal{B}^+$ for at least $(\alpha/2)L$ values of j, and is in $X \sim \bigcup \mathcal{B}$ for less than $(4\delta/\alpha)L$ values of j, $1 \leqq j \leqq L$.

Now we partition X_0 by a $\mathcal{P}_1^{NL}$-measurable partition. The elements of the partition are characterized by

(a) an integer $k = 0, \cdots, N-1$,

(b) a choice of a subset J of $\{1, \cdots, L\}$ containing $(\alpha/2)L$ elements (use rational α and choose L so $(\alpha/2)L$ is an integer),

(c) for each $j \in J$, a member of $\bigcup \mathcal{B}^+$,

(d) for each $j \notin J$, a member of $\mathcal{B} \cup \{X \sim \mathcal{B}\}$.

The corresponding subset of X_0 is those x such that T^{k+jN} lies in the corresponding set designated in (c) and (d), $j = 1, \cdots, L$. These subsets of X_0 are not actually disjoint, but they cover X_0, and we simply disjointify them in some way. The sets of this partition of X_0 have $d_{NL}^{\mathcal{P}}$-diameter no greater than $(1-4\delta/\alpha)r + 4\delta/\alpha$.

We now estimate the cardinality of this partition, or rather its log divided by LN: this is dominated by

$$\frac{\log N}{NL}+\frac{\log\left(\begin{matrix}L\\ [\alpha L]+1\end{matrix}\right)}{NL}+\frac{\alpha}{4}\frac{\log|\mathscr{B}_+|}{N}+\left(1-\frac{\alpha}{4}\right)\frac{\log(|\mathscr{B}|+1)}{N}$$

(assuming $\alpha<\frac{1}{2}$). The first two terms may be made arbitrarily small (for any fixed N) be choosing L large. Also, for large N, the last term differs by an arbitrarily small amount from $(1-\alpha)\log|\mathscr{B}|/N$. Since $|\mathscr{B}_+|\leqq 2^{N(b-\delta)}$ while $|\mathscr{B}|\leqq 2^{N(b+\delta)}$, what we are left with is dominated by $b-\alpha^2/4+\delta(1-\alpha/4)$. Thus, choosing δ small gives our partition cardinality $\leqq 2^{LN(b-\alpha^2/8)}$. But its sets have $d^{\mathscr{P}}_{NL}$ diameters $<(1-4\delta/\alpha)r+4\delta/\alpha$. In view of the continuity of r-entropy (Proposition 2.3) we have arrived at a contradiction. □

Now, the stronger version, replacing $\log|\mathscr{B}|$ by $H(\mathscr{B})$:

2.5. Theorem. *Let T be ergodic on (X,μ). Let $\mathscr{P}$ be a partition and $r>0$. Then for any $\varepsilon>0$ there exist N_0 and $\delta>0$ such that if $N>N_0$ and $\mathscr{B}$ is a $(\mathscr{P},N,r)$ family of measure $>1-\delta$ satisfying $H(\mathscr{B})/N<k_r(T,\mathscr{P})+\delta$, then the B in $\mathscr{B}$ such that*

$$k_r(T,\mathscr{P})-\varepsilon<-\frac{\log\mu(B)}{N}<k_r(T,\mathscr{P})+\varepsilon$$

form a set of total measure $\geqq 1-\varepsilon$.

Proof. Let $\mathscr{B}_+$ be those $B\in\mathscr{B}$ with $\mu(B)>2^{-N(b-\alpha)}$, where α is fixed, and $b=k_r(T,\mathscr{P})$. Now take a $(\mathscr{P},N,r)$ family $\mathscr{A}$ of measure $>1-\delta$ and with $|\mathscr{A}|<2^{N(b+\delta)}$. Let $\mathscr{B}'=\mathscr{B}_+\cup\{A\sim\bigcup\mathscr{B}_+:A\in\mathscr{A}\}$. Then $\mathscr{B}'$ is a $(\mathscr{P},N,r)$ family of measure $>1-\delta$, and $|\mathscr{B}'|<|\mathscr{A}|+|\mathscr{B}_+|$. But clearly $|\mathscr{B}_+|<2^{Nb}$, so if $\delta<\alpha$ we have $|\mathscr{B}'|<2^{Nb}+2^{N(b+\delta)}<2^{N(b+2\delta)}$ if N is sufficiently big. By choosing δ small enough for Proposition 2.3 to come into effect, we can guarantee that, after removal of a subfamily of $\mathscr{B}'$ of arbitrarily small measure, the remaining $B'\in\mathscr{B}'$ all have measure $<2^{-N(b-\alpha)}$, for any preassigned α. In particular, this holds for $\mathscr{B}_+$, and hence for $\mathscr{B}$. Therefore, without loss of generality, we may assume that, for all $B\in\mathscr{B}$, $\mu(B)<2^{-N(b-\alpha)}$, where α is preassigned. Let $\mathscr{B}_-=$ those B for which $\mu(B)<2^{-N(b+\beta)}$. Then, setting $c=\mu(\bigcup\mathscr{B}_-)$ and $d=\mu(\bigcup(\mathscr{B}\sim\mathscr{B}_-))$, we have $H(\mathscr{B})/N\geqq d(b-\alpha)+c(b+\beta)$. Now, $c+d\geqq 1-\delta$, so:

$$H(\mathscr{B})\geqq(1-\delta)b-d\alpha+c\beta\geqq(1-\delta)b-\alpha+c(\beta-\alpha).$$

If c can be kept bounded away from 0 for arbitrarily large N and small δ, then by choosing α very small and then δ very small, we contradict the assumption that $H(\mathscr{B})/N<b+\delta$.

Thus, $\mu(\bigcup \mathcal{B}_-)$ must go to 0 as N gets big and δ small, and the proof is complete. □

2.6. COROLLARY. *For all* $r \geqq 0$, $k_r(T, \mathcal{P}) = h_r(T, \mathcal{P})$.

2.7. COROLLARY. *In the definitions of* $h_r(T, \mathcal{P})$ *and* $k_r(T, \mathcal{P})$, "$\exists N_0$ *such that for all* $N > N_0$" *may be replaced by "for infinitely many* N".

2.8. COROLLARY. *Given* $r > 0$, $\varepsilon > 0$ *and* $\alpha > 0$, *then there exist* N_0 *and* $\delta > 0$ *such that if* $\mathcal{B}$ *is a* $(\mathcal{P}, N, r)$ *family of measure* $> \alpha$, $N > N_0$, *and* $H(\mathcal{B})/\alpha N < h_r(T, \mathcal{P}) + \delta$, *then those* $B \in \mathcal{B}$ *with* $h_r(T, \mathcal{P}) - \varepsilon < -\log \mu(B)/N < h_r(T, \mathcal{P}) + \varepsilon$ *form a set of measure* $> \alpha - \varepsilon$.

The argument for this is close to Theorem 2.6, so we omit it.

2.9. COROLLARY. *Given* $r > 0$, $\varepsilon > 0$ *and* $\alpha > 0$ *then for sufficiently large* N, *any* $(\mathcal{P}, N, r)$ *family* $\mathcal{B}$ *such that each* $B \in \mathcal{B}$ *has measure* $> 2^{-N(h_r(T,\mathcal{P})-\varepsilon)}$, *must satisfy* $\mu(\bigcup \mathcal{B}) < \varepsilon$.

2.10. REMARK. The foregoing ideas could be used to define other "approximate entropies." For example: suppose instead of using *all* sets of diameter r, we only permit $(\mathcal{P}, N, r)$ families $\mathcal{B}$ whose members B have names which are subsets of *spheres of radius* $r/2$ in the set of all $N - \mathcal{P}$ names. Since the extra conditions we have imposed are preserved under taking subsets and under concatenation, we get a new (perhaps larger) r-entropy for which all the results of this section hold. It would be interesting to know the relation between the new r-entropy and the old one.

3. r-entropy and equipartition in $\mathbf{R}^n$

There is very little difficulty in extending the ideas and results of the previous section to actions of $\mathbf{Z}^n \times \mathbf{R}^n$. The main change is to substitute normalized Haar measure on generalized intervals for normalized counting measure in defining the metric on strings. Indeed, every one of the results of the previous section carries over, with only notational changes, although a couple of the proofs will require some work, some of which will be carried out below. To avoid unnecessary notational complication, we shall here discuss only the case of $\mathbf{R}^n$, which will be used in Sections 4 and 5.

C_N will always denote the cube whose vertices have all coordinates at 0 or N, N being a positive real number. If f and g are measurable functions from a measurable subset C of $\mathbf{R}^n$ to a finite index set, then we denote by $d_C(f, g)$, or —

when no confusion is possible — $d(f, g)$, the number $(1/|C|)|\{f \neq g\}|$, where $|C|$ means the Haar measure of C. For a measure-preserving action ϕ of $\mathbf{R}^n$ and partition $\mathcal{P}$, we can define $d_C^{\mathcal{P}}(x, y) = d(f, g)$, where $f(v) = \mathcal{P}(\phi_v x)$ and $g(v) = \mathcal{P}(\phi_v y)$, $v \in C$; that is, $(1/|C|)|\{v : \mathcal{P}_v(x) \neq \mathcal{P}_v(y)\}|$. Denote by $\mathcal{P}_C$ the σ-field spanned by $\{\phi_v^{-1}\mathcal{P} : v \in C\}$. Then a family $\mathcal{B}$ of disjoint sets will be called a $(\mathcal{P}, N, r)$ family if

(1) each $B \in \mathcal{B}$ is in $\mathcal{P}_{C_N}$,

(2) each $B \in \mathcal{B}$ has $d_{C_N}^{\mathcal{P}}$-diameter $\leqq r$.

The distance thus defined will be treated rather casually from the notational point of view, but we believe this will cause no difficulty.

$h_r(\phi, \mathcal{P})$ and $k_r(\phi, \mathcal{P})$ may be defined in obvious analogy with Section 2. Of the properties listed (a) $\cdots$ (e) in that section, all but (a) are immediate. We must prove, then, that $\lim_{r\to 0} k_r(\phi, \mathcal{P}) = h_0(\phi, \mathcal{P})$. The definition of $h_0(\phi, \mathcal{P})$ may be taken as

$$h_0(\phi, \mathcal{P}) = \lim_{D \downarrow 0} |C_D|^{-1} h_0(\phi^D, \mathcal{P}),$$

where ϕ^D is the $\mathbf{Z}^n$-action obtained from ϕ on the $D\mathbf{Z}^n$-lattice: $\phi_v^D = \phi_{Dv}$. (The entropy of a $\mathbf{Z}^n$ process is defined in [1] and [4], and is completely analogous to that of a $\mathbf{Z}$ action, i.e., a transformation.)

3.1. Theorem. *For any ergodic probability-preserving* $\mathbf{R}^n$ *action* ϕ, *we have*

$$\lim_{r\to 0} k_r(\phi, \mathcal{P}) = h_0(\phi, \mathcal{P}).$$

Proof. In the course of this proof we shall use analogs of some of the simpler results of Section 2 for actions of $\mathbf{Z}^n$. The problems introduced by these generalizations are only notational.

Fix $D > 0$. The *continuous* $d^{\mathcal{P}}$ distance on C_N between x and y may be computed by taking the *discrete* $d^{\mathcal{P}}$ distance between $\phi_v x$ and $\phi_v y$ over the C_D lattice points in C_N, and taking the normalized integral of this as v ranges over C_D (provided N/D is an integer).

Suppose $\mathcal{B}$ is a $(\mathcal{P}, N, \delta)$ family of measure $1 - \varepsilon$. If x and y are in the same B, $d_{C_N}(x, y) < \delta$. Then a sequence of Fubini theorem estimates tells us that for any preassigned $\varepsilon > 0$, a small enough choice of δ will guarantee that there exist a set $V \subset C_D$ with $|V|/|C_D| > 1 - \varepsilon$, for each $v \in V$ a set $S_v \subset X$ with $\mu(S) > 1 - \varepsilon$, and for each $B \in \mathcal{B}$ and $x \in B \cap S_v$ a set R_x with $\mu(R_x)/\mu(B) > 1 - \varepsilon$, such that if $v \in V$, $x \in B \cap S_v$, and $y \in R_x$ then the discrete distance from $\phi_v x$ to $\phi_v y$ (over the C_D lattice points in C_N) is less than $\varepsilon/2$. Thus if y', y'' are in R_x the discrete distance from $\phi_v y'$ to $\phi_v y''$ is less than ε. Choose some fixed $v \in V$ and some $x(B)$ is each nonempty $B \cap S_v$, and let $\mathcal{B}_0 = \{\phi_{-v} R_{x(B)} : B \cap S_v \neq \emptyset\}$. Then $\mu(\bigcup \mathcal{B}_0) > 1 - 3\varepsilon$, $|\mathcal{B}_0| \leqq |\mathcal{B}|$, and each $B \in \mathcal{B}_0$ has discrete $d^{\mathcal{P}}$ diameter $< 3\varepsilon$ over the C_D lattice points in C_N. Expand each set $B \in \mathcal{B}_0$ by throwing in the set $\tilde{B}$ of all points which have the same $\mathcal{P}$ name over the C_D lattice in C_N as any point of B. The family so obtained consists of sets which are measurable with respect to $\bigvee \{\phi_{-v}\mathcal{P} : v \in D\mathbf{Z}^n \cap C_N\}$. However, they may no longer be disjoint. Disjointify them. Thus we have produced a $(\mathcal{P}, N/D, 3\varepsilon)$ family for ϕ^D, of measure $> 1 - 3\varepsilon$, and of cardinality $\leqq |\mathcal{B}|$. Since ε was completely arbitrary, this shows that

$$\lim_{\delta \to 0} k_\delta(\phi, \mathcal{P}) \geqq |C_D|^{-1} h_0(\phi^D, \mathcal{P}),$$

which gives the result, in one direction.

To go in the opposite direction, we first single out a lemma.

3.2. LEMMA. *Fix $\varepsilon > 0$. There exist $D > 0$ and N_0 so that if $N > N_0$, and if we let L_x be the set of C_D lattice points v in C_N for which*

$$|\{w \in C_D : \mathcal{P}(\phi_{w+v} x) = \mathcal{P}(\phi_v x)\}| < (1 - \varepsilon)|C_D|,$$

then $\{x : |L_x| > (1 - \varepsilon)(N/D)^n\}$ has measure $> 1 - \varepsilon$. ($(N/D)^n$ is of course just the number of C_D lattice points in C_N.)

SKETCH OF PROOF. By a straightforward argument involving Fubini's Theorem, the Lebesgue Continuity Theorem, and stationarity, we get

$$|\{\omega \in C_D : \mathcal{P}(\phi_\omega x) = \mathcal{P}(x)\}| > (1 - \varepsilon)|C_D|$$

for all x in a set of measure $> 1 - \varepsilon^2$. From here the result is easy.

PROOF OF THEOREM 3.1. Let R be the set of x of measure $> 1 - \varepsilon$ in the statement of Lemma 3.2. Then R is a $\mathcal{P}_{C_N}$-measurable set, and if B is a $\{\phi_v \mathcal{P} : v \in D\mathbf{Z}^n \cap C_N\}$-measurable set of discrete diameter $\leqq r$ for the $\{(\phi_v^D, \mathcal{P}) : v \in \mathbf{Z}^n \cap C_N\}$ process, then $B \cap R$ is $\mathcal{P}_{C_N}$-measurable and has continuous diameter $\leqq r + 2\varepsilon$ for the $\{(\phi_v, \mathcal{P}) : v \in C_N\}$ process. Thus

$$k_{r+2\varepsilon}(\phi, \mathcal{P}) \leqq |C_D|^{-1} k_r(\phi^D, \mathcal{P}).$$

Fixing D and letting $r \to 0$ gives

$$k_{2\varepsilon}(\phi, \mathcal{P}) \leqq |C_D|^{-1} h_0(\phi^D, \mathcal{P}).$$

Let $\varepsilon \downarrow 0$; and $D \downarrow 0$ (as forced by ε). Then we get $\lim_{\varepsilon \to 0} k_\varepsilon(\phi, \mathcal{P}) \leqq h_0(\phi, \mathcal{P})$. □

As for the remaining results of Section 2, the proofs all go over with only notational changes. The only exception to this is Proposition 2.4, where the continuity provides an added complication in making the count at the end of the argument. This is handled by using a fine lattice and the Lebesgue continuity theorem.

Here is one final small result which will be used in Section 5.

3.2. Proposition. *Given any $S \subset X$ and $0 < \varepsilon < \mu(S)$, and any point r, then for any sufficiently big N there is a $(\mathcal{P}, N, r)$ family $\mathcal{A}$ which covers S up to measure ε, and a special point $x(A) \in A \cap S$ for each $A \in \mathcal{A}$, with the $x(A)$ at mutual $d^{\mathcal{P}}_{C_N}$ distance $> r/2$, with $|\mathcal{A}| > 2^{|C_n|(k_r(\phi,\mathcal{P})-\varepsilon)}$, and with $\mu(A) < 2^{-|C_N|(k_r(\phi,\mathcal{P})-\varepsilon)}$ for all $A \in \mathcal{A}$.*

Proof. Choose a maximal family $x_1, \cdots, x_l$ of points in S which are mutually more than $r/2$ apart in the $d^{\mathcal{P}}_{C_N}$ metric. Let B_i be a sphere of radius $r/2$ around x_i in this metric. Then $\bigcup_i B_i \supset S$, because of maximality of $\{x_1, \cdots, x_l\}$. Disjointify the B_i, getting a family $\{B'_1, \cdots, B'_l\}$. Since no x_i is in B_j if $j \neq i$, it follows that $x_i \in B'_i$. Reject all B'_i of measure $> 2^{-N(b-\varepsilon/2)}$, where $b = k_r(\phi, \mathcal{P})$. If N is sufficiently large, then — by the $\mathbf{R}^n$ version of Corollary 2.9 — we are rejecting a set of measure $< \varepsilon$. Also, the remaining B'_i, since they cover S up to ε and each has measure $\leqq 2^{-N(b-\varepsilon/2)}$, must be in number greater than $(\mu(S) - \varepsilon)2^{N(b-\varepsilon/2)}$. Choose N so large that $(\mu(S) - \varepsilon)2^{N\varepsilon/2} > 1$. This does it. □

3.3. Remark. It is important to realize that all these notions are unaffected by isomorphism. This is not quite as empty a remark for flows or $\mathbf{R}^n$ actions as it would be for transformations. Let ϕ on (X, μ) and $\bar{\phi}$ on $(\bar{X}, \bar{\mu})$ be isomorphic $\mathbf{R}^n$ actions. That is, let f be an a.e. defined 1–1 m.p. map: $X \to \bar{X}$. Then $(v, x) \mapsto (v, f(x))$ is jointly measurable, and measure-preserving for the product measures, so given any partition $\mathcal{P}$ on X and letting $\bar{\mathcal{P}} = f(\mathcal{P})$, we have that the functions $v \mapsto \mathcal{P}(\phi_v x)$ and $v \mapsto \bar{\mathcal{P}}(\bar{\phi}_v f(x))$ agree a.e. as functions of v, for a.e. x, so isomorphism don't change anything.

4. $\bar{d}$ joinings and semifinitely determined actions

We begin with a discussion of the $\bar{d}$ metric.

Let (C, λ) be a Lebesgue space with finite measure, and let $\xi = \{\xi_v, v \in C\}$ and $\eta = \{\eta_v, v \in C\}$ be stochastic processes so that the functions $(v, x) \mapsto \xi_v(x)$ and $(v, y) \mapsto \eta_v(y)$ are jointly measurable, with values in the same finite set. A *joining* of ξ and η is a stochastic process $(\bar{\xi},\bar{\eta})=\{(\bar{\xi}_v,\bar{\eta}_v):v\in C\}$ such that $\bar{\xi}\approx\xi$ and $\bar{\eta}\approx\eta$. Associated with each joining is a number, which we call its *gap*: the infimum of $\{b:\{z:\lambda\{v:\bar{\xi}_v(z)\neq\bar{\eta}_v(z)\}\geqq b\lambda(C)\}$ has measure $\geqq b\}$. $\bar{d}(\xi,\eta)$ is the infimum of the gaps of all possible joinings. Thus $\bar{d}(\xi,\eta)<b \Leftrightarrow$ there exists a joining for which $\lambda\{v:\bar{\xi}_v(z)\neq\bar{\eta}_v(z)\}<b\lambda(C)$ except for a set of z of measure $<b$. Two joinings $(\bar{\xi},\bar{\eta})$ and $(\hat{\xi},\hat{\eta})$ are said to be *equivalent* if they have the same joint distribution as stochastic processes.

4.1. Remark. Any joining of ξ and η may be realized in certain canonical way. Let $(X,\mathcal{F},\mu)$ be the measure space of ξ, let $\mathcal{F}_0$ be the σ-subfield of $\mathcal{F}$ generated by ξ, let $\hat{X}$ be the space of atoms of $\mathcal{F}_0$ in X, and let $(\hat{\mathcal{F}}_0,\hat{\mu})$ be the images of $\mathcal{F}_0$ and μ in $\hat{X}$. Similarly if η is defined on $(Y,\mathcal{G},\nu)$, and $\mathcal{G}_0$ the σ-subfield generated by η, we get $(\hat{Y},\hat{\mathcal{G}}_0,\hat{\nu})$. Let $\hat{\xi}$ and $\hat{\eta}$ be the images of ξ and η on $\hat{X}$ and $\hat{Y}$ respectively. Then if we are given *any* joining of $\bar{\xi}$ and $\bar{\eta}$ as described above, the joint distribution may in a straightforward way be transferred to $(\hat{X}\times\hat{Y},\hat{\mathcal{F}}_0\times\hat{\mathcal{G}}_0)$, so that we get a measure ρ on $\hat{\mathcal{F}}_0\times\hat{\mathcal{G}}_0$ which projects to $\hat{\mu}$ and $\hat{\nu}$ respectively. This special joining might reasonably be called a *minimal* joining; so to every joining there corresponds an equivalent *minimal* joining on $\hat{X}\times\hat{Y}$.

4.2. Remark. There is a subtlety which we should elucidate here. While our joining carries processes isomorphic to $(\xi,\mathcal{F}_0,\mu\mid\mathcal{F}_0)$ and $(\eta,\mathcal{G}_0,\mu\mid\mathcal{G}_0)$, it does *not* necessarily have a σ-subfield corresponding to all of $\mathcal{F}$ or all of $\mathcal{G}$. We shall now arrange matters to provide these. We shall build fibres, which will be measure spaces, over each point $(\hat{x},\hat{y})$ of our minimal joining. Let p be the projection: $X\to\hat{X}$ and $q:Y\to\hat{Y}$. Then $(X,\mathcal{F},\mu)$ may be fibred over $(\hat{X},\hat{\mathcal{F}}_0,\hat{\mu})$, getting measure spaces $(\pi^{-1}(\hat{x}),\mathcal{F}_{\hat{x}},\mu_{\hat{x}})$ for each $\hat{x}$. Similarly, $(Y,\mathcal{G},\nu)$ may be fibred to get $(\theta^{-1}(\hat{y}),\mathcal{G}_{\hat{y}},\nu_{\hat{y}})$. Then we make a new measure space which is fibred over $(\hat{X}\times\hat{Y},\hat{\mathcal{F}}_0\times\hat{\mathcal{G}}_0,\hat{\mu}\times\hat{\nu})$ by putting the fibre $(\pi^{-1}(\hat{x})\times\theta^{-1}(\hat{y}),\ \mathcal{F}_{\hat{x}}\times\mathcal{G}_{\hat{y}},\ \mu_{\hat{x}}\times\nu_{\hat{y}})$ over $(\hat{x},\hat{y})$. This space may then be regarded as the product of $(X,\mathcal{F})$ and $(Y,\mathcal{G})$, and its measure projects onto μ and ν; furthermore, the process $(\hat{\xi}\circ\pi,\hat{\eta}\circ\theta)$ is isomorphic as a joint process to $(\hat{\xi},\hat{\eta})$. Such a joining we call a *comprehensive* joining. Now we define $\bar{d}_{C_N}((\phi,\mathcal{P}),(\psi,\mathcal{Q}))$ by using the processes $\phi_{-v}\mathcal{P}$ and $\psi_{-v}\mathcal{Q}$, $v\in C_N$; and

$$\bar{d}((\phi,\mathcal{P}),(\psi,\mathcal{Q}))=\limsup_{N\to\infty}\bar{d}_{C_N}((\phi,\mathcal{P}),(\psi,\mathcal{Q}));$$

which, because of stationarity, is really just $\lim_{N\to\infty}\bar{d}_{C_N}((\phi,\mathcal{P}),(\psi,\mathcal{Q}))$. The joinings involved may always be assumed to be comprehensive in the sense of 4.2.

4.3. PROPOSITION. *Given a process $(\phi, \mathcal{P})$, a finite set $V \subset \mathbf{R}^n$, and $\varepsilon > 0$, and setting $\mathcal{R} = \vee\{\phi_{-v}\mathcal{P} : v \in V\}$, then there exist $\delta > 0$ and N_0 such that if $N > N_0$ and $\bar{d}_{C_N}((\phi, \mathcal{P}), (\phi', \mathcal{P})) < \delta$ then there is a joining of gap $< \varepsilon$ for which the $\mathcal{R}'$ corresponding to $\mathcal{R}$ satisfies $|\mathcal{R} - \mathcal{R}'| < \varepsilon$.*

PROOF. Assume V is contained in a cube of side M centered at the origin. Then for any joining of $(\phi, \mathcal{P})$ with $(\phi', \mathcal{P}')$, and $x \in X$ and $x' \in X'$,

$$|\{v \in C_N : \mathcal{R}(x) \neq \mathcal{R}'(x')\}| \leqq M + |V|\,|\{v \in C_N : \mathcal{P}(x) \neq \mathcal{P}'(x')\}|.$$

Thus, $\bar{d}_{C_N}((\phi, \mathcal{R}), (\phi', \mathcal{R}')) \leqq M/N + |V|\,\bar{d}_{C_N}((\phi, \mathcal{P}), (\phi', \mathcal{P}'))$. Therefore w.l.o.g. we may assume $\mathcal{R} = \mathcal{P}$.

If the joining has gap $< \delta$, then an application of Fubini's theorem tells us that, if ρ is the measure on the space of the joining, then

$$\rho\{(x, x') : \mathcal{R}(\phi_v x) \neq \mathcal{R}'(\phi'_v x')\} < \delta,$$

except for a set of v in C_N of proportion $< \delta$.

Let $D = (\delta)^{1/n}N$. Then there must be some $v_0 \in C_D$ satisfying $\rho\{(x, x') : \mathcal{R}(\phi_{v_0} x) \neq \mathcal{R}'(\phi'_{v_0} x')\} < \delta$. Let the joining be the comprehensive one of Remark 4.2, on the space $X \times X'$. Then $\phi_{v_0} \times \phi'_{v_0}$ acts; it sends ρ to a new measure, which is still a joining; and since the shift was by a vector in C_D, there is not much change in the gap: for

$$|\{v : \mathcal{R}(\phi_{v+v_0} x) \neq \mathcal{R}'(\phi'_{v+v_0} x'), v \in C_N\}|$$

is no greater than $|\{v : \mathcal{R}(\phi_v x) \neq \mathcal{R}(\phi'_v x'), v \in C_N\}| + |\{v : v + v_0 \in C_N, v \notin C_N\}|$, and $|\{v : v + v_0 \in C_N, v \notin C_N\}| < |C_N| - |C_{N-D}|$. Thus, the translation of ρ by $\phi_{v_0} \times \phi'_{v_0}$ gives a joining of gap less than

$$\delta + \frac{|C_N| - |C_{N-D}|}{|C_N|} = \delta + (1 - (1 - \delta)^{1/n})^n,$$

which is small if δ is small; and in the new joining, $\{(x, x') : (x) \neq \mathcal{R}'(x')\}$ has measure $< \delta$. □

4.4. PROPOSITION. *Given $(\phi, \mathcal{P})$, ε, and V as before, and defining $\mathcal{R}$ as before, then $\exists \delta$ and N so that if $\bar{d}_{C_N}((\phi, \mathcal{P}), (\phi', \mathcal{P}')) < \delta$ then there is another joining, of gap $< \varepsilon$ such that, when regarded as a joining of the* discrete *processes $(\psi, \mathcal{R})$ and $(\psi', \mathcal{R}')$ obtained by restrictions ϕ and ϕ' to $\mathbf{Z}^n$, then this discrete joining also has gap $< \varepsilon$.*

PROOF. This goes along similar lines. Again we may assume $\mathcal{P} = \mathcal{R}$. But we use Fubini's theorem on the given joining of gap $< \delta$ to conclude that, except for a set of (x, x') of measure $< \delta$, the set of v in C, for which $\{w : \mathcal{R}(\phi_{w+v}x) \neq \mathcal{R}'(\phi'_{w+v}x')$ for a proportion more than $\delta(|C_N| - |C_1|)/|C_N|$ of the integral lattice points w in $C_N\}$ must have measure $< \delta$. Again, choose such a non-exceptional v_0; then, for any of the non-exceptional (x, x'), we have $\mathcal{R}(\phi_{w+v_0}x) = \mathcal{R}'(\phi'_{w+v_0}x')$ except for a set of lattice points w of small proportion. As before, shift the measure of the joining by $\phi_{v_0} \times \phi'_{v_0}$. This changes the gap of the continuous joining by very little if N is large, since $v_0 \in C_1$; and now we have $\mathcal{R}(\phi_w x) = \mathcal{R}(\phi'_w x')$ for a large proportion of lattice points w, for most (x, x'). This completes the proof. □

Now we recall the notion of "finitely determined" (FD). A **Z** process $(T, \mathcal{P})$ is called FD if, given $\varepsilon > 0$, there exist γ, δ, and N such that if $(\bar{T}, \bar{\mathcal{P}})$ is another process satisfying

(1) $h_0(\bar{T}, \bar{\mathcal{P}}) > h_0(T, \mathcal{P}) - \gamma$,

(2) $|\text{dist} \bigvee_{j=1}^{N} T^{-j}\mathcal{P} - \text{dist} \bigvee_{j=1}^{N} \bar{T}^{-j}\bar{\mathcal{P}}| < \delta$,

then $\bar{d}((\bar{T}, \bar{\mathcal{P}}), (T, \mathcal{P})) < \varepsilon$.

Ornstein [6] showed that T is a Bernoulli transformation $\Leftrightarrow$ there is a generator $\mathcal{P}$ such that $(T, \mathcal{P})$ is FD $\Leftrightarrow$ for every $\mathcal{Q}$, $(T, \mathcal{Q})$ is FD. The notion clearly extends to actions of $\mathbf{Z}^n$, as does the aforementioned result of Ornstein (see [4]). Furthermore, if in (2) we replace

$$\left| \text{dist} \bigvee_{j=1}^{N} T^{-j}\mathcal{P} - \text{dist} \bigvee_{j=1}^{N} \bar{T}^{-j}\bar{\mathcal{P}} \right| \quad \text{by} \quad \bar{d}_N((T, \mathcal{P}), (\bar{T}, \bar{\mathcal{P}})),$$

which gives an equivalent definition, then in this form the definition has an obvious meaning for flows, or for $\mathbf{R}^n$ actions. Can it be used to characterize those $(\phi, \mathcal{P})$ for which ϕ is a Bernoulli action? The answer is a strong *no*: if ϕ is a flow, then no $(\phi, \mathcal{P})$ can be finitely determined. The basic reason for this is that a flow can have a generating partition which is extremely close to the trivial partiton, so that entropy is extremely sensitive to small $\bar{d}$ changes.

There is a substitute definition. It makes sense for transformations equally well, and it may be shown that this new definition will work as well as FD to characterize $(T, \mathcal{P})$ with T Bernoulli. However, we shall discuss only the case of $\mathbf{R}^n$ actions. The idea is to use approximate entropy, which for fixed r changes continuously in the $\bar{d}$ metric.

4.5. Definition. Let ϕ be an aperiodic ergodic action of $\mathbf{R}^n$ and $\mathcal{P}$ a partition such that $(\phi, \mathcal{P})$ has finite entropy. We say that $(\phi, \mathcal{P})$ is *semifinitely determined* (SFD) if given $\varepsilon > 0$ $\exists \gamma > 0$ and for each positive r some δ_r and N_r such that if $(\bar{\phi}, \bar{\mathcal{P}})$ is another aperiodic ergodic $\mathbf{R}^n$ process and satisfies

(1) $h_r(\bar{\phi}, \bar{\mathscr{P}}) > h_0(\phi, \mathscr{P}) - \gamma$,
(2) $\bar{d}_{C_{N_r}}((\bar{\phi}, \bar{\mathscr{P}}), (\phi, \mathscr{P})) < \delta_r$,
then $\bar{d}((\bar{\phi}, \bar{\mathscr{P}}), (\phi, \mathscr{P})) < \varepsilon$.

4.6. REMARK. The definition makes sense for actions of $\mathbf{Z}$ or $\mathbf{Z}^n$ as well, and in these cases it may be shown that a process is SFD if and only if it is FD.

4.7. THEOREM. *Let ϕ be an aperiodic and ergodic action of $\mathbf{R}^n$, $\mathscr{P}$ a generator, and suppose that for some generator $\mathscr{Q}$ under the $\mathbf{Z}^n$ action ψ obtained from ϕ by restriction to the integral lattice, the process $(\psi, \mathscr{Q})$ is* FD. *Then $(\phi, \mathscr{P})$ is* SFD.

PROOF. First choose a D so small that there is a set $S \subset X$ of measure $> 1 - \varepsilon/100$ such that for any $x \in S$ a fraction $> 1 - \varepsilon/100$ of the C_D lattice points v in C_1 have the following property:

$$|\{w \in C_D : \mathscr{P}(\phi_{w+v}x) \neq \mathscr{P}(\phi_v x)\}| < \frac{\varepsilon}{100} |C_D|.$$

Now take a process $(\bar{\phi}, \bar{\mathscr{P}})$ with $\bar{d}_{C_{N_1}}((\bar{\phi}, \bar{\mathscr{P}}), (\phi, \mathscr{P})) < \delta_1$. I claim that if δ_1 is small enough and N_1 big enough, then the same state of affairs holds, but with $\varepsilon/10$ replacing $\varepsilon/100$.

To see this: let V be the set of C_D lattice points in C_1, and let $\mathscr{R} = \bigvee \{\phi_{-v}\mathscr{P} : v \in V\}$. Choose δ_1 and N_1 as in Proposition 4.4, but for $(\varepsilon/100)^3$. Then if $\bar{d}_{C_{N_1}}((\phi, \mathscr{P}), (\bar{\phi}, \bar{\mathscr{P}})) < \delta_1$, we have:

(1) except for a set of $(x, \bar{x})$ of measure $< (\varepsilon/100)^2$,

$$|\{v \in C_{N_1} : \mathscr{P}(\phi_v x) \neq \bar{\mathscr{P}}(\bar{\phi}_v \bar{x})\}| < \left(\frac{\varepsilon}{100}\right)^3 |C_N|,$$

(2) except for a set of $(x, \bar{x})$ of measure $< (\varepsilon/100)^3$,

$$|\{v \in \mathbf{Z}^n \cap C_{N_1} : \mathscr{R}(\phi_v x) \neq \bar{\mathscr{R}}(\bar{\phi}_v \bar{x})\}| < \left(\frac{\varepsilon}{100}\right)^3 |\mathbf{Z}^n \cap C_{N_1}|.$$

Let T be the union of these two exceptional sets.

If $(x, \bar{x}) \notin T$, then there is a set of integral lattice points v, of proportion $> 1 - \varepsilon/100$, so that both $|\{w \in C_1 : \mathscr{P}(\phi_{w+v}x) \neq \bar{\mathscr{P}}(\bar{\phi}_{w+v}\bar{x})\}| < (\varepsilon/100)^2$ and $\mathscr{R}(\phi_v x) = \bar{\mathscr{R}}(\bar{\phi}_v \bar{x})$. Thus, there is some integral lattice point v_0 such that, for all $(x, \bar{x})$ not in some exceptional set U of measure $< \varepsilon/100 + 2(\varepsilon/100)^2$, $\mathscr{R}(\phi_{v_0}x) = \bar{\mathscr{R}}(\bar{\phi}_{v_0}\bar{x})$ and

$$|\{w \in C_1 : \mathscr{P}(\phi_{w+v_0}x) \neq \bar{\mathscr{P}}(\bar{\phi}_{w+v_0}\bar{x})\}| < \left(\frac{\varepsilon}{100}\right)^2.$$

The set $\bar{S}$ of $\bar{x}$ for which there is some $x \in S$ with $(\phi_{-v_0}x, \phi_{-v_0}\bar{x}) \notin U$, has measure $> 1 - \varepsilon/100 - \varepsilon/100 > 1 - \varepsilon/10$. Let $\bar{x}$ be in $\bar{S}$, so we have $x \in S$ with $(x, \bar{x}) \notin U$. Then $\bar{\mathscr{R}}(\bar{x}) = \mathscr{R}(x)$, so $\bar{\mathscr{P}}(\bar{\phi}_v\bar{x}) = \mathscr{P}(\phi_v x)$ for all $v \in V$.

$$|\{v : \bar{\mathscr{P}}(\bar{\phi}_v\bar{x}) \neq \mathscr{P}(\phi_v x), v \in C_1\}| < \left(\frac{\varepsilon}{100}\right)^2.$$

Finally, $\{w : \mathscr{P}(\phi_{w+v}x) \neq \mathscr{P}(\phi_v x), w \in C_D\}| < (\varepsilon/100)|C_D|$, except for a fraction of $v \in V$ smaller than $\varepsilon/100$. Combining these, we see that, for such $(x, \bar{x})$, except for a set of $v \in V$ of proportion $< \varepsilon/50$, we have both

$$|\{w \in C_D : \mathscr{P}(\phi_{w+v}x) \neq \mathscr{P}(\phi_v x)\}| < \frac{\varepsilon}{100}|C_D| \quad \text{and} \quad |\{w \in C_D : \bar{\mathscr{P}}(\bar{\phi}_{w+v}\bar{x}) \neq \mathscr{P}(\phi_{w+v}x)\}| < \frac{\varepsilon}{100}|C_D|,$$

and since $\mathscr{P}(\phi_v x) = \bar{\mathscr{P}}(\bar{\phi}_v\bar{x})$ for $x \in V$, we get: except for a set of $v \in V$ of proportion $< \varepsilon/50$, $|\{w \in C_D : \bar{\mathscr{P}}(\bar{\phi}_{w+v}\bar{x}) \neq \bar{\mathscr{P}}(\bar{\phi}_v\bar{x})\}| < (\varepsilon/50)|C_D|$.

Now choose some large L.

We will attempt to construct a $\bar{\mathscr{Q}}$ in $\bar{X}$ satisfying

$$\text{(i)} \qquad \bar{d}((\psi, \mathscr{Q}), (\bar{\psi}, \bar{\mathscr{Q}})) < \varepsilon/10L.$$

We first show how this may be used to get $\bar{d}((\phi, \mathscr{P}), (\bar{\phi}, \bar{\mathscr{P}})) < \varepsilon$.

Choose an L so large that if we set $\mathscr{S} = \bigvee\{\psi_{-v}\mathscr{Q} : v \in \mathbf{Z}^n, \|v\| < L\}$ (where $\|v\| =$ max of coordinates), then $\exists \mathscr{R}_0 < \mathscr{S}$ with $|\mathscr{R}_0 - \mathscr{R}| < \varepsilon/10$. I claim that by choosing δ_1 yet smaller and N_1 yet bigger, we can get, for the corresponding partition $\bar{\mathscr{S}}$ on $\bar{X}$ and the corresponding $\bar{\mathscr{R}}_0$, $|\bar{\mathscr{R}}_0 - \bar{\mathscr{R}}| < \varepsilon/10$. This is immediate from Proposition 4.3.

Notation. $C'_N = \{v \in \mathbf{R}^n = \|v\| \leqq N\}$. So $|C'_N| = (2N)^n$, while $|C_N| = N^n$.

Take a measure on $X \times \bar{X}$ which gives a comprehensive joining of $(\psi, \mathscr{Q})$ and $(\bar{\psi}, \bar{\mathscr{Q}})$ on C_M of gap $< \varepsilon/10L$. (M will be chosen very large.) Then the set of pairs $(x, \bar{x}')$ for which

$$\{v : \mathscr{Q}(\phi_v x) \neq \bar{\mathscr{Q}}(\bar{\phi}_v\bar{x}), v \in C_M \cap \mathbf{Z}^n\}| < \frac{\varepsilon}{10L} M^n$$

has measure $> 1 - \varepsilon/10L$. If $(x, \bar{x})$ is in this set, then

$$|\{v : \mathscr{S}(\phi_v x) \neq \bar{\mathscr{S}}(\bar{\phi}_v\bar{x}), v \in W\}| < \frac{\varepsilon}{10} M^n$$

where W is the integral points of C_M with a slab of thickness L removed from the outside. Since $\mathscr{R}_0 \subset \mathscr{S}$ and $\bar{\mathscr{R}}_0 \subset \bar{\mathscr{S}}$, we likewise have for such $(x, \bar{x})$,

$$|\{v : \mathscr{R}_0(\phi_v x) \neq \bar{\mathscr{R}}_0(\bar{\phi}_v \bar{x}), v \in W\}| < \frac{\varepsilon}{10} M^n$$

$< (\varepsilon/9)|M - 2L|^n$ if M is sufficiently big. Now, by the ergodic theorem, if M is sufficiently big we can exclude sets of measure $< \varepsilon/10$ from X and $\bar{X}$ and get, on the remaining points,

$$|\{v : \mathscr{R}(\phi_v x) \neq \mathscr{R}_0(\phi_v x), v \in W\}| < \frac{\varepsilon}{9}(M - L)^n,$$

$$|\{v : \bar{\mathscr{R}}(\bar{\phi}_v \bar{x}) \neq \bar{\mathscr{R}}_0(\bar{\phi}_v \bar{x}), v \in W\}| < \frac{\varepsilon}{9}(M - L)^n.$$

Thus, except for a set of $(x, \bar{x})$ of measure $< \varepsilon/3$, we have

$$|\{v : \mathscr{R}(\phi_v x) \neq \bar{\mathscr{R}}(\bar{\phi}_v \bar{x}), v \in W\}| < \frac{\varepsilon}{3} M^n.$$

but if $\mathscr{R}(\phi_v x) = \bar{\mathscr{R}}(\phi_v \bar{x})$ and $x \in S$ and $\bar{x} \in \bar{S}$,

$$|\{w : \mathscr{P}(\phi_{w+v} x) \neq \bar{\mathscr{P}}(\bar{\phi}_{w+v} \bar{x}), w \in C_1\}| < \frac{\varepsilon}{10} + \frac{\varepsilon}{100}.$$

Thus,

$$|\{u : \mathscr{P}(\phi_u x) \neq \bar{\mathscr{P}}(\bar{\phi}_u \bar{x}), u \in C_M\}| < \left(\frac{\varepsilon}{3} + \frac{\varepsilon}{10} + \frac{\varepsilon}{100}\right) M^n$$

$$+ \text{the volume of a boundary of thickness } L,$$

which, if M is big enough, is less than εM^n. Furthermore, the exceptional $(x, \bar{x})$ form a set of measure $< \varepsilon$. Thus $\bar{d}_{C_M}((\phi, \mathscr{P}), (\bar{\phi}, \bar{\mathscr{P}})) < \varepsilon$.

It remains to produce $\bar{\mathscr{Q}}$ satisfying (i) $\bar{d}((\psi, \mathscr{Q}), (\bar{\psi}, \bar{\mathscr{Q}})) < \varepsilon/10L$. Since $(\psi, \mathscr{Q})$ is FD, there exist γ_0, δ_0 and N_0 so that if $(\bar{\psi}, \bar{\mathscr{Q}})$ satisfy

(a) $h_0(\bar{\psi}, \bar{\mathscr{Q}}) > h_0(\psi, \mathscr{Q}) - \gamma_0$,

(b) $(\bar{\psi}, \bar{\mathscr{Q}})$ is closer than δ_0 to $(\psi, \mathscr{Q})$ in the $\bar{d}$ distance on $C_{N_0} \in \mathbf{Z}^n$,

then $(\bar{\psi}, \bar{\mathscr{Q}})$ satisfies (i). Thus the problem is to achieve (a) and (b).

First we try for (b). Choose some finite set $W \subset \mathbf{R}^n$ such that, setting $\mathscr{T} = \vee \{\phi_{-v}\mathscr{P} : v \in W\}$, then there is some $\mathscr{Q}_0 \subset \mathscr{T}$ with $|\mathscr{Q}_0 - \mathscr{Q}| < \delta_0/100N_0$. Let $\bar{\mathscr{T}}$ correspond to $\mathscr{T}$ and $\bar{\mathscr{Q}}$ to $\mathscr{Q}_0$ for the $(\bar{\phi}, \bar{\mathscr{P}})$ process. Now choose δ_1 so small and N_1 so big that $\bar{d}_{C_{N_1}}((\phi, \mathscr{P}), (\bar{\phi}, \bar{\mathscr{P}})) < \delta_1$ implies $\bar{d}_{N_0}((\psi, \mathscr{T}), (\bar{\psi}, \bar{\mathscr{T}})) < \delta_0/2$. This may be done, by Proposition 4.4. Then also $d_{N_0}((\psi, \mathscr{Q}_0), (\bar{\psi}, \bar{\mathscr{Q}})) < \delta_0/2$. But then $d_{N_0}((\psi, \mathscr{Q}), (\bar{\psi}, \bar{\mathscr{Q}})) < \delta_0$.

Finally, we try for (a). It is here that the r-entropy assumption will come into play. Suppose $h_r(\bar{\phi}, \bar{\mathscr{P}}) > h_0(\phi, \mathscr{P}) - \gamma_0 = h_0(\psi, \mathscr{Q}) - \gamma_0$. We need to get $h_0(\bar{\psi}, \bar{\mathscr{Q}}) \geqq h_r(\bar{\phi}, \bar{\mathscr{P}})$. Choose K (depending on r) so that the $\mathscr{Q}$ name on $C'_K \cap \mathbf{Z}^n$

determines a set of $\bar{d}$ diameter $< r/4$ for $\mathcal{P}$ on C_1, except for a set R of measure $< \varepsilon_1$ (ε_1 to be chosen later). Choose $|\mathcal{Q}_0 - \mathcal{Q}|$ yet smaller, $< \varepsilon_1^2/|C'_K|$. (This affects the size of the set W of the previous paragraph, and hence forces the δ_1 and N_1 of that discussion to depend on r.) For all x outside of a set of measure $< \varepsilon_1^2$, the $\mathcal{Q}_0$ name on $C_K \cap \mathbf{Z}^n$ is the same as the $\mathcal{Q}$ name. Thus, for all x outside of a set of measure $< \varepsilon_1^2$, the $\mathcal{Q}_0$ name determines a set of $\bar{d}$ diameter $< r/4$ for $\mathcal{P}$ on C_1. Proposition 4.4 may be applied to tell us that if $\bar{d}_{C_N}((\phi, \mathcal{P}), (\bar{\phi}, \bar{\mathcal{P}}))$ is small enough, for large enough N, then the same will hold for $(\bar{\phi}, \bar{\mathcal{P}})$ if we replace $r/4$ by $r/2$ and ε_1 by $2\varepsilon_1$. That is, outside of a set E of measure $< (2\varepsilon_1)^2$, the $\bar{\mathcal{Q}}$ name on $C'_K \cap \mathbf{Z}^m$ determines a set of $\bar{d}^{\bar{\mathcal{P}}}_{C_1}$-radius $< r/2$. The argument is much like the previous use of Proposition 4.4 in this theorem.

Now apply the ergodic theorem to $\bar{\psi}$: if M is sufficiently large (in particular, $M \gg K$) then, with the exception of a set of $\bar{x}$ of measure $< 3\varepsilon_1$, $\bar{\phi}_v\bar{x}$ will be in E for a fraction less than $3\varepsilon_1$ of the v in $C_M \cap \mathbf{Z}^m$. Thus, among the non-exceptional x, the $\mathcal{Q}$ name on $C_M \cap \mathbf{Z}^m$ determines a set of $d^{\bar{\mathcal{P}}}_{C_M}$ diameter $< r/2 + 4\varepsilon_1$. Choose $\varepsilon_1 < r/8$, so the $d^{\bar{\mathcal{P}}}_{C_M}$ diameter is $< r$. Now, applying the Equipartition Theorem to $(\bar{\psi}, \bar{\mathcal{Q}})$, we see that $h_0(\bar{\psi}, \bar{\mathcal{Q}}) \geqq h_r(\bar{\phi}, \bar{\mathcal{P}})$. □

5. The isomorphism theorem in $\mathbf{R}^n$

Let ϕ be an aperiodic and ergodic action of $\mathbf{R}^n$ and $\mathcal{P}$ a partition so that $(\phi, \mathcal{P})$ has finite entropy.

5.1. THEOREM. *Suppose $\mathcal{P}$ is a generator under ϕ, $(\phi, \mathcal{P})$ is* SFD, *and ψ is an aperiodic ergodic action of $\mathbf{R}^n$ with $h(\psi) = h(\phi) < \infty$. Then there is a partition $\mathcal{Q}$ so that $(\psi, \mathcal{Q}) \approx (\phi, \mathcal{P})$.*

As in [6], it suffices to show the following.

5.2. FUNDAMENTAL LEMMA. *If $(\phi, \mathcal{P})$ is* SFD, *ψ is aperiodic and ergodic, $h(\phi, \mathcal{P}) = h(\psi)$, and $\mathcal{R}$ satisfies $\bar{d}((\psi, \mathcal{R}), (\phi, \mathcal{P})) < (\varepsilon/100)^3$, then for any $\varepsilon' > 0$ $\exists \mathcal{R}'$ with $\bar{d}((\psi, \mathcal{R}'), (\phi, \mathcal{P})) < \varepsilon'$ and $|\mathcal{R}' - \mathcal{R}| < \varepsilon$.*

This will itself be proved in several steps. For a set $S \subset X$ and $C \subset \mathbf{R}^n$, CS will mean $\bigcup\{\phi_v(S) : v \in C\}$. It will be convenient to always take K, L, M, N, etc. to be integers.

5.3. LEMMA. *Given M, $r > 0$, and $\varepsilon > 0$, there exists N_0 and δ such that if $E = C_N F$ is a Rokhlin tower made with the N cube, $N > N_0$, and with error $< \delta$, and $\mathcal{H}$ is the partition consisting of $X \sim E$ together with the sets $\{CF : C$ a cube from the $1/M$-lattice in the N-cube$\}$, then $h_r(\psi, \mathcal{H}) < \varepsilon$.*

PROOF. Choose L so big that translation of $C_{1/M}$ by any vector in $C_{1/L}$ moves a fraction less that $r/4$ of its volume outside of itself. Choose $\delta < r/4$ (for starters). Let N be any positive integer. Choose $\alpha > 0$, and choose K so large that — by the ergodic theorem — except for a set of y of measure $< \alpha$, $|\{v \in C_{KN} : \phi_v y \in E\}|/(KN)^n > 1 - 2\delta$.

Now, for each K, we concentrate on the y's in this "good" set, call it S, and show that if δ is sufficiently small and K sufficiently big then S may be covered by few enough $\mathcal{H}_{C_{KN}}$-measurable sets of $\bar{d}_{C_{KN}}$-radius $< r$ to get the desired entropy estimate.

$C_{KN}y$ is a copy of C_{KN}, and is thus divided up into K^n cubes of side N; call these *fixed* cubes. Cubes (or parts of cubes) of the form $C_N y'$, $y' \in F$, are scattered over it; call these *random* cubes. They are of course disjoint. At most $2nK^{n-1}$ random cubes can intersect the boundary of $C_{KN}y$, and if K is sufficiently large they constitute a proportion less than δ of the volume of $C_{KN}y$. So $C_{KN}y$ is covered up to proportion 3δ by random cubes lying entirely inside it.

No fixed cube can contain more than one point of F, but at least $K^n(1-3\delta)$ of them contain one, by the previous sentence. Now divide each fixed cube into "little cubes" of side $1/LM$, and number them in some systematic way. To each $y \in S$ assign the following data:

(1) the set of fixed cubes containing a point of F (making some sort of convention about boundary occurrences),

(2) for each fixed cube containing a point of F, the number of the little cube containing the point of F (once again, making a convention about boundaries).

If the same data is assigned to y_1 and y_2, then, by the choice of L, the $\mathcal{H}_{C_{KN}}$-names of y_1 and y_2 agree up to $4\delta < r$. This gives a partition of S into $\mathcal{H}_{C_{KN}}$-measurable sets of radius $< r$, and we must count them. An upper bound is $\sum_{j=[K^n(1-3\delta)]}^{K}\binom{K^n}{j}(LMN)^{jn}$. Observing that $K^n - [K^n(1-3\delta)] \leqq 3K^n\delta$, and that

$$\binom{K^n}{k} = \binom{K^n}{K^n - k} \qquad \text{for any } k,$$

we see that the sum is maximized by

$$([3K^n\delta]+1)\binom{K^n}{[3K^n\delta]}(LMN)^{K^n n}, \qquad \text{provided } 3\delta < \frac{1}{2}.$$

Taking logs and dividing by $(KN)^n$, we get a sum of three terms:

(a) $\dfrac{1}{(KN)^n}\log([eK^n\delta]+1) \leqq \dfrac{1}{K^n}\log 4K^n \to 0 \quad \text{as } K \to \infty,$

(b) $\dfrac{\log\binom{K^n}{[3K^n\delta]}}{(KN)^n} \leqq \dfrac{\log\binom{K^n}{[3K^n\delta]}}{K^n} \to 3\delta\log 3\delta + (1-3\delta)\log(1-3\delta) \text{ as } K \to \infty,$

(c) $\frac{n \log LMN}{N^m} \to 0$ as $N \to \infty$.

Choice of N sufficiently large and δ sufficiently small gives the desired result. □

The next lemma is a "strong" Rokhlin theorem. The construction is just an adaptation of the usual one for $\mathbf{Z}$, as in [8].

5.4. Lemma. *Given any partition $\mathcal{Q}$ and N, δ there exists a Rokhlin tower $E = C_N F$ with error $< \delta$ such that, letting π be the projection from E onto F, and $\rho = \nu \circ \pi^{-1}$ normalized, then the process $\{\mathcal{Q}_v, v \in C_N\}$ on (F, ρ) given by $\tilde{\mathcal{Q}}_v = \psi_v^{-1}(\mathcal{Q}_v \mid \psi_v F)$ is within δ of the process $\{\mathcal{Q}_v : v \in C_N\}$ in the $\bar{d}$ metric.*

Proof. Let $E' = C_{KN}F'$ be a Rokhlin tower with error δ'. Divide the points of F' into a family $\mathcal{S}$ of sets of diameter $< \delta'$ for the $\bar{d}$ metric on $\mathcal{Q}_{C_{KN}}$; there are only finitely many of these. Let π' be the projection from E' onto F'. Then $\nu \circ \pi'^{-1}$ is a continuous measure. Normalize it to get a measure ρ'. Choose a random variable $\theta : F' \to C_N$ which is uniformly distributed on each S, with respect to ρ'. Now define a new tower with base $F = \bigcup_{y \in F'} \bigcup_v \psi_{\theta(y)} y$, where v ranges over a set of centers for the lattice of N-cubes inside the KN-cube, with the bordering N-cubes omitted. Let $E = C_N F$. It is reasonably clear that for sufficiently large K and small δ', this Rokhlin tower will have the desired property. □

5.5. Lemma. *For any fixed N_0, and a.e. x, the empirical distribution of the $\mathcal{P}_{C_{N_0}}$ name on the $\mathcal{P}_{C_N}$ name of x converges in the $\bar{d}$ metric to the distribution of $\mathcal{P}_{C_{N_0}}$, as $N \to \infty$.*

Proof. First choose a $(\mathcal{P}, N_0, \varepsilon_0)$ family $\mathcal{A}_0$, with ε_0 very small, which covers all but ε_0 of the space. Then by the ergodic theorem, the emprical distribution of the finite partition $\mathcal{A}_0$ over $C_N x$, with N large, will be $\bar{d}$ close to its true distribution.

Thus a measure-preserving bijection may be set up between the space of ϕ and the cube C_N with normalized Lebesgue measure, such that if x is outside of a certain set of measure $< \varepsilon$, then for all v ouside of a set of measure $< \varepsilon_0$, the name of $C_v x$ will be precisely that of the point which corresponds to v. This correspondence gives a $\bar{d}$ joining to within $2\varepsilon_0$. □

5.6. Proof of the Fundamental Lemma. Let γ', δ'_r and N'_r be the quantities provided by the definition of SFD for $(\phi, \mathcal{P})$ and ε'. Then what is required is to construct $\mathcal{R}'$ and find $r > 0$ such that

(i) $|\mathscr{R}' - \mathscr{R}| < \varepsilon$,

(ii) $h_r(\psi, \mathscr{R}') > h(\phi, \mathscr{P}) - \gamma'$,

(iii) $\bar{d}_{C_{N'}}((\psi, \mathscr{R}'), (\phi, \mathscr{P})) < \delta'_r$.

Choose $\mathscr{Q} \supset \mathscr{R}$ so $h(\phi, \mathscr{Q}) > h(\phi, \mathscr{P}) - \gamma'/2$. Choose $s \ll \varepsilon$ and so $h_{4s}(\phi, \mathscr{Q}) > h(\phi, \mathscr{P}) - \gamma'/2$. Since $h_{4s}(\phi, \mathscr{Q}) < h(\phi, \mathscr{Q}) \leqq h(\phi, \mathscr{P})$, choose $r \ll \varepsilon$ so $h_{8r}(\phi, \mathscr{P}) > h_{4s}(\phi, \mathscr{Q})$.

Since $\bar{d}((\psi, \mathscr{R}), (\phi, \mathscr{P})) < (\varepsilon/100)^2$, choose such a joining of the processes; in its weakest form, this means that for each N we can find a space $(\bar{X}, \bar{\mu})$ and processes $\{\bar{\mathscr{R}}_v, \bar{\mathscr{P}}_v, v \in C_N\}$, with $\{\bar{\mathscr{R}}_v, v \in C_N\} \approx \{\mathscr{R}_v, v \in C_N\}$, $\{\bar{\mathscr{P}}_v, v \in C_N\} \approx \{\mathscr{P}_v, v \in C_N\}$, and

$$\frac{1}{N^n} |\{v : \bar{\mathscr{R}}_v(\bar{x}) \neq \bar{\mathscr{P}}_v(\bar{x})\}| < \left(\frac{\varepsilon}{100}\right)^2$$

except on a set of measure $< (\varepsilon/100)^2$.

We may assume further that there is a partition $\bar{\mathscr{Q}}_v \supset \bar{\mathscr{R}}_v$, $v \in C_N$, with

$$\{\bar{\mathscr{Q}}_v, v \in C_N\} \approx \{\mathscr{Q}_v, v \in C_N\}.$$

This last may be achieved by putting in the $\bar{\mathscr{Q}}$ process independently of the $\bar{\mathscr{P}}$ process on each fibre of the $\bar{\mathscr{R}}$ process.

Now take N at least large enough that there is a $(\mathscr{Q}, N, s)$ family $\mathscr{B}$ of total measure $\leqq 1 - \delta_0$ with $|\mathscr{B}| \geqq 2^{N^n(b-\delta_0)}$ and each $B \in \mathscr{B}$ of measure $\geqq 2^{-N^n(a+\delta_0)}$, where $b = h_s(\psi, \mathscr{Q})$. This may be done for any preassigned δ_0, by the r-equipartition theorem of Section 3. We may also assume that on $\bigcup \mathscr{B}$ a shift by ψ_v with $v \in C_{\delta_1}$ will change the $\mathscr{Q}_{C_N}$ name of a point by less than $s/2$ in $\bar{d}$; this is an application of the ergodic theorem and the Lebesgue theorem on continuity of translation.

Next, choose a $(\mathscr{P}, N, 8r)$ family $\mathscr{A}$ of measure $\geqq 1 - \delta_0$ with $|\mathscr{A}| \geqq 2^{N^n(a-\delta_0)}$ and each $A \in \mathscr{A}$ of measure $\leqq 2^{-N^n(a-\delta_0)}$, where $a = h_{8r}(\phi, \mathscr{P})$. We may also assume that $\bigcup \mathscr{A}$ consists entirely of points x for which the empirical distribution of $\mathscr{P}_{C_{N'}}$ on $C_N x$ differs in $\bar{d}$ from the true distribution of $\mathscr{P}_{C_{N'}}$ by less than δ_0, and that a shift by ψ_v with $v \in C_{\delta_1}$ will change the $\mathscr{P}_{C_N}$ name of x by less than $r/2$ in $\bar{d}$. Finally, we may assume, by Proposition 3.2, that there are special points $x(A) \in A \in \mathscr{A}$ with mutual distance $\geqq 4r$.

Now transfer the whole picture to $\bar{X}$, getting families $\bar{\mathscr{B}}$, $\bar{\mathscr{A}}$. If δ_0 has been chosen small enough, so that $\bar{\mathscr{B}}$ and $\bar{\mathscr{A}}$ nearly fill $\bar{X}$, then more than $1 - \varepsilon/50$ of X will be filled by sets $\bar{\mathscr{B}}$ in $\bar{\mathscr{B}}$... call them the *good* ones ... for which a proportion of at least $1 - \varepsilon/50$ of the measure of $\bar{B}$ is occupied by sets $\bar{A}$ in $\bar{\mathscr{A}}$ such that $\bar{A} \cap \bar{B}$ contains points for which the $\bar{\mathscr{P}}_{C_N}$ name and the $\bar{\mathscr{R}}_{C_N}$ name differ in $\bar{d}$ by less than $\varepsilon/100$; the facts that $r \ll \varepsilon$ and $s \ll \varepsilon$ then tell us that the

$\bar{\mathcal{P}}_{C_N}$ name of $\bar{x}(\bar{A})$ and the $\bar{\mathcal{R}}_{C_N}$ name of any point in this $\bar{B}$ differ in $\bar{d}$ by less than $\varepsilon/50$. Call such $\bar{B}$ and $\bar{A}$ *compatible.*

Now let $c = a - b$, and choose $\delta_0 < c/3$. Then each $\bar{\mathcal{A}}$ set has measure $\leqq 2^{-N^n(a-\delta_0)} \leqq 2^{-N^n(b+2\delta_0)} = 2^{-N^n\delta_0}2^{-N^n(b+\delta_0)}$. Choosing N large makes this $<(1-\varepsilon/50)2^{-N^n(b+\delta_0)}$. Thus, each collection of k good $\bar{\mathcal{B}}$ sets must intersect at least k compatible $\bar{\mathcal{A}}$ sets. By the Marriage Lemma, we assign to each $\bar{\mathcal{B}}$ set a different compatible $\bar{\mathcal{A}}$ set; to the remaining $\bar{\mathcal{B}}$ sets we assign any of the remaining $\bar{\mathcal{A}}$ sets (since there are plenty left, by our estimate).

Now we transfer the entire picture to the space Y. First we build a Rokhlin tower $E = C_NF$ of error δ and, as in Lemma 4.2, such that if π is the projection: $E \to F$, and ρ is $\nu \cdot \pi^{-1}$ normalized, and $\tilde{\mathcal{Q}}_v = \pi(\mathcal{Q} \mid \phi_v F)$, then

$$\bar{d}(\{\tilde{\mathcal{Q}}_v : v \in C_N\}, \{\mathcal{Q}_n : v \in C_N\}) < \delta_1,$$

where δ_1 is chosen after δ_0 and is *really* small.

Now, since δ_1 is so small, we can copy $\bar{\mathcal{B}}$ into (F, ρ); that is, we get a family $\tilde{\mathcal{B}}$ of disjoint $\{\tilde{\mathcal{Q}}_v : v \in C_N\}$ measurable sets covering F up to, say, $2\delta_0$, each having $\bar{d}$ diameter $< 2s$, each $\tilde{\mathcal{B}}$ set $\tilde{B}$ corresponding to a certain $\bar{B}$ in $\bar{\mathcal{B}}$, and — aside from a total number of points of total measure $< \delta_0$ — the $\{\tilde{\mathcal{Q}}_v, v \in C_N\}$ name of each point of $\tilde{B}$ being within δ_0 of the $\tilde{\mathcal{Q}}_{C_N}$ name of some point of $\bar{B}$ in the d metric. This depends only on the smallness of δ_1: the choice of δ_1 is not affected by making N larger of δ smaller.

Now define the partition $\mathcal{R}'$, indexed like $\mathcal{P}$, as follows. Let F_0 be the union of the sets in $\tilde{\mathcal{B}}$, and let $E_0 = \pi^{-1}(F_0)$. Thus, assuming $\delta < \delta_0$, E_0 has measure $> 1 - 3\delta_0$. If $y \in E_0$ then $y = \psi_u y_0$ for some y_0 in some $\tilde{B}$. The corresponding $\bar{B}$ was assigned some $\bar{A} \in \bar{\mathcal{A}}$ by the Marriage Theorem. We assign to y_0 as its $\mathcal{R}'_{C_N}$ name the $\bar{\mathcal{P}}_{C_N}$ name of the special point $x(A)$ in A, where A is the isomorph of $\bar{A}$ in $\mathcal{A}$. Thus, the index of the $\mathcal{R}'$ set containing y is the same as the index of the $\bar{\mathcal{P}}_u$ set containing $\bar{x}(\bar{A})$. Outside E_0, define the partition $\mathcal{K}'$ arbitrarily.

Now we must show that (i), (ii), and (ii) will be satisfied by $\mathcal{R}'$ if the parameters have been chosen properly.

(i) $|\mathcal{R}' - \mathcal{R}| < \varepsilon$.

This is because if a point y of F_0 lies in $\tilde{B}$, then its $\mathcal{R}_{C_N}$ name is δ_0 close in $\bar{d}$ to some $\bar{\mathcal{R}}_{C_N}$ name in $\bar{B}$, which in turn is within $\varepsilon/100$ of any $\bar{\mathcal{P}}_{C_N}$ name in the corresponding $\bar{A}$; but these are almost all the same as the $\mathcal{P}_{C_N}$ names in the corresponding A. Now, the $\mathcal{R}'_{C_N}$ name of y is one of these $\bar{\mathcal{P}}_{C_N}$ names: specifically, the name of the *special* point. Thus, the $\bar{d}_{C_N}$ distance of the $\mathcal{R}'_{C_N}$ and the $\mathcal{R}_{C_N}$ names of points in F_0 is less than $\delta_0 + \varepsilon/100$. So $|\mathcal{R}'|E_0 - \mathcal{R}|E_0| < \delta_0 + \varepsilon/100$, and since $\mu(E_0) > 1 - (\delta_0 + \varepsilon/100)$ if δ is small enough, (i) is proven.

(ii) $h_r(\psi, \mathcal{R}') > h(\phi, \mathcal{P}) - \gamma'$.

Choose $M > 0$ and use N, δ, and M as in Lemma 5.3 to make a "height" partition $\mathcal{H}$ with respect to the tower $C_N F$. Choose M so large and a number r_0 so small that, if $\bar{N}$ is sufficiently large, then knowledge of the $\mathcal{H}$ name to within r_0 in $d_{\bar{N}}$ on $C_{\bar{N}}y$ tells the position of occurrences of E_0 columns in $C_{\bar{N}}y$ to within an error of less than δ_1, for a proportion at least $1-\delta$, of these occurrences. Then choose N so large and δ so small that $h_{r_0}(\psi, \mathcal{H}) < \gamma'/8$, and choose $\bar{N}$ so large that there is an $(\mathcal{H}, C_{\bar{N}}, r_0)$ family $\mathcal{G}$ of measure $> 1-\delta_2$ with $|\mathcal{G}| < 2^{\bar{N}''\gamma'/2}$. Also choose any $(\mathcal{R}', \bar{N}, r)$ family $\mathcal{D}$ of measure $> 1-\delta_2$. Then $|\mathcal{D} \vee \mathcal{G}| < |\mathcal{D}| \cdot 2^{\bar{N}\gamma'/2}$.

Now, choice of a $D \cap G$ in $\mathcal{D} \vee \mathcal{G}$ tells us the position of a proportion $1-\delta_1$ of the E_0 columns, to within δ_1. Thus, since a change of position by less than δ_1 changes the $\bar{\mathcal{P}}_{C_N}$ names of the special points by less than $r/2$, and these are the same as the $\mathcal{R}'$ names along the E_0 columns, it follows that these $\mathcal{R}'$ names remain at distance at least $3r$ apart. Now, a set in $\mathcal{D}$ has diameter $\leqq r$ for the $\bar{d}$ metric on $\mathcal{P}_{C_{\bar{N}}}$ names. Then for each point y in some $D \cap G$ we can determine, for each E_0 column which lies fully within $C_{\bar{N}}y$, which $\pi^{-1}B$ that column belongs to. Then we know the $\mathcal{Q}$ name on this E_0 column, to within $4s$. But of course we know the position of that E_0 column only to within a "shove" of δ_1. If we then consider a y such that $C_{\bar{N}}y$ consists at least $1-\delta_1$ of such E_0 columns, then, taking account of the fact that a δ_1-shove on a $\mathcal{Q}$ name in an E_0 column moves it by $< s/2$, the $\mathcal{Q}_{C_{\bar{N}}}$ name of this y is known to within $3s + \delta_1 < 4s$. Since the set of y omitted can be made arbitrarily small by choosing $\bar{N}$ big and δ_2 small, it follows that

$$|\mathcal{D}| \geqq 2^{-\bar{N}\gamma'/2} 2^{\bar{N}b},$$

or

$$\frac{1}{\bar{N}} \log |\mathcal{D}| \geqq h_{4s}(\psi, \mathcal{Q}) - \frac{\gamma'}{2} > h(\phi, \mathcal{P}) - \gamma'.$$

Thus $h_r(\psi, \mathcal{R}') > h(\phi, \mathcal{P}) - \gamma'$. It is important to note that N and δ are still free to be made, respectively, larger and smaller.

(iii) Finally, to achieve

$$d_{C_{N'}}((\psi, \mathcal{R}'), (\phi, \mathcal{P})) < \delta'_r,$$

we simply observe that, by choosing N large, we can make the empirical distribution of $\mathcal{P}_{C_{N'}}$ names on the $\mathcal{P}_{C_N}$ name of each special point as close in $\bar{d}$ as desired to the true distribution of $\mathcal{P}_{C_{N'}}$; consequently, on E_0, the distribution of $\mathcal{R}'_{C_{N'}}$ is as close in $\bar{d}$ as desired to that of $\mathcal{P}_{C_{N'}}$. But $\nu(E_0) > 1 - 3\delta_0$, and δ_0 can be chosen after r, so we are done. □

A more careful application of the fundamental lemma gives:

5.7. COROLLARY. *Suppose $\mathcal{P}$ is a generator under ϕ, and $(\phi, \mathcal{P})$ is* SFD. *Then given $\varepsilon > 0$ there is a γ, N_r, and δ_r such that if $h(\psi) = h(\psi, \mathcal{P})$ and $\mathcal{R}$ is a partition satisfying*

(i) $h_r(\psi, \mathcal{R}) > h(\phi, \mathcal{P}) - \gamma$,

(ii) $\bar{d}_{N_r}((\psi, \mathcal{R}), (\phi, \mathcal{P})) < \delta_r$,

then there is a partition $\mathcal{R}'$ with $|\mathcal{R} - \mathcal{R}'| < \varepsilon$ and $(\psi, \mathcal{R}') \approx (\phi, \mathcal{R})$.

5.8. REMARK. Theorem 4.1 could have been proven assuming $h(\psi) \geqq h(\phi)$, rather than equality. Similarly for the above corollary. One method would be to build a factor of ψ with entropy $= h(\phi)$, and apply our results to this factor. Another would be to increase the entropy of ϕ to that of ψ by taking its Cartesian product with some SFD action of the proper entropy, and showing that the product is again SFD. However, since the present form suffices for the isomorphism theorem, we won't carry out either of these arguments.

5.9. THEOREM. *Suppose $\mathcal{P}$ is a generator for the* SFD *action ϕ, and similarly $\mathcal{Q}$ for the* SFD *action ψ, and both have equal entropy. Then $\phi \approx \psi$.*

First we isolate an improved "copying" lemma.

5.10. LEMMA. *Let $(\phi, \mathcal{Q}) \approx (\hat{\phi}, \hat{\mathcal{Q}})$ be aperiodic and ergodic processes on X and $\hat{X}$, and let $\mathcal{P}$ be a partition on X. Then for any M, finite set $U \subset C_M$, and $\varepsilon > 0$ there is a partition $\hat{\mathcal{P}}$ of $\hat{x}$ so that $|(\mathcal{P} \vee \mathcal{Q})_U - (\hat{P} \vee \hat{\mathcal{Q}})_U| < \varepsilon$ and $\bar{d}_{C_M}((\phi, \mathcal{P} \vee \mathcal{Q}), (\hat{\phi}, \hat{\mathcal{P}} \vee \hat{\mathcal{Q}})) < \varepsilon$.*

PROOF. Let $E = C_L F$ be a Rokhlin tower for ϕ, for very big L, and small error δ. Let $\mathcal{A}$ be a partition of F into a $(\mathcal{Q}, L, \delta)$ family. Let $\hat{E} = C_L\hat{F}$ be the corresponding tower for $\hat{\phi}$, and let $\hat{\mathcal{A}}$ be the corresponding partition. Next choose a partition of F into a $(\mathcal{P}, L, \delta)$ family $\mathcal{B}$. Make a partition $\hat{\mathcal{B}}$ of F so that $\mathcal{A} \vee \mathcal{B}$ and $\hat{\mathcal{A}} \vee \hat{\mathcal{B}}$ have the same distribution. Choose a point in each $A \cap B$, and give each point of $\hat{A} \cap \hat{B}$ a $\hat{\mathcal{P}}_{C_L}$ name which agrees with the $\mathcal{P}_{C_L}$ name of the chosen point in $A \cap B$. Assign the partition $\hat{\mathcal{P}}$ arbitrarily outside $\hat{E}$. Finally, define a map: $X \to \hat{X}$ by mapping each $A \cap B$ to $\hat{A} \cap \hat{B}$ in a 1-1 m.p. fashion, and $X \sim E$ to $\hat{X} \sim \hat{E}$ in a m.p. fashion. If L is big enough and δ small enough, this map will be a $\bar{d}_{C_M}$ match to within ε between $(\phi, \mathcal{P} \vee \mathcal{Q})$ and $(\hat{\phi}, \hat{\mathcal{P}} \vee \hat{\mathcal{Q}})$.

Now apply Lemma 4.3 to get the rest. □

Next, a lemma like that on p. 17 of [6], but without the assumption that $\mathcal{P}$ generates under ϕ restricted to $\mathbf{Z}$. Theorem 5.9 will then follow from 5.11 exactly as in the discrete case.

5.11. Lemma. *Let $\mathscr{P}$ be a generator for ϕ, with $(\phi, \mathscr{P})$ SFD. Let $(\phi, \mathscr{Q})$ have full entropy, and also be SFD. Then for any $\varepsilon > 0$ $\exists \bar{\mathscr{Q}}$ such that*

(1) $(\phi, \bar{\mathscr{Q}}) \approx (\phi, \mathscr{Q})$,

(2) $|\mathscr{Q} - \bar{\mathscr{Q}}| < \varepsilon$,

(3) $\mathscr{P} \overset{\varepsilon}{\subset} \bar{\mathscr{Q}}_{\mathbf{R}^n}$.

Proof. Choose a finite set $V \subset \mathbf{R}^n$ so that $\mathscr{P}_V \supset \mathscr{Q}_0$ with $|\mathscr{Q}_0 - \mathscr{Q}| < \varepsilon/8$. Let $|V| = M$.

Next apply Corollary 5.7 to $(\phi, \mathscr{P})$ to get γ, N_r and δ_r for $\varepsilon/4M$. Choose r so $h_r(\phi, \mathscr{Q}) > h(\phi, \mathscr{Q}) - \gamma/2$. Choose $\bar{\varepsilon}$ so that if $|\mathscr{Q}^* - \mathscr{Q}| < \bar{\varepsilon}$ then $h_r(\phi, \mathscr{Q}^*) > h_r(\phi, \mathscr{Q}) - \gamma/2$.

Next apply the previous lemma: choose K so $V \subset C_K$, and $K > N_r$, and so $\exists \mathscr{Q}_1 \subset \mathscr{P}_{C_K}$ with $|\mathscr{Q}_1 - \mathscr{Q}| < \bar{\varepsilon}$. Now build a partition $\mathscr{P} \subset \mathscr{Q}_{\mathbf{R}^m}$ so that $(\phi, \mathscr{P} \vee \mathscr{Q})$ and $(\phi, \mathscr{P}' \vee \mathscr{Q})$ match to within δ_r in $\bar{d}_{C_K}$, while $(\mathscr{P} \vee \mathscr{Q})_V$ and $(\mathscr{P}' \vee \mathscr{Q})_V$ have distribution distance so close that if we set $\mathscr{Q}_0'$ equal to the partition in $\mathscr{P}_V'$ built like the partition $\mathscr{Q}_0$ in $\mathscr{P}_V$, then $|\mathscr{Q}_0' - \mathscr{Q}| < |\mathscr{Q}_0 - \mathscr{Q}| + \varepsilon/8$. Thus $\mathscr{Q}_0' \subset \mathscr{P}_V'$ and $|\mathscr{Q}_0' - \mathscr{Q}| < \varepsilon/4$.

Now, $h_r(\phi, \mathscr{P}') \geqq h_r(\phi, \mathscr{Q}_1) \geqq h_r(\phi, \mathscr{Q}) - \gamma/2 \geqq h(\phi, \mathscr{Q}) - \gamma = h(\phi, \mathscr{P}) - \gamma$. Since also the $\bar{d}$ distance from $\mathscr{P}_{C_K}'$ to $\mathscr{P}_{C_K}$ is $< \delta_r$, there is some $\bar{\mathscr{P}} \subset \mathscr{Q}_{\mathbf{R}^n}$ with $|\bar{\mathscr{P}} - \mathscr{P}'| < \varepsilon/4M$ and $(\phi, \bar{\mathscr{P}}) \approx (\phi, \mathscr{P})$. Let $\bar{\mathscr{Q}}_0$ be constructed from $\bar{\mathscr{P}}_V$ in the same way in which $\mathscr{Q}_0$ and $\mathscr{Q}_0'$ are constructed from $\mathscr{P}_V$ and $\mathscr{P}_V'$. Then also $\bar{\mathscr{Q}}_0$ is the image of $\mathscr{Q}_0$ in the isomorphism from $(\phi, \mathscr{P})$ to $(\phi, \bar{\mathscr{P}})$. Then $|\mathscr{Q}_0' - \bar{\mathscr{Q}}_0| < \varepsilon/4$.

Now we reverse roles. Choose a finite set $W \subset \mathbf{R}^n$ so that $\mathscr{Q}_W \supset \bar{\mathscr{P}}_0$ with $|\bar{\mathscr{P}}_0 - \bar{\mathscr{P}}| < \varepsilon/8$. Let $|W| = N$.

Again, apply Corollary 4.7, to $(\phi, \mathscr{Q})$ this time, and for $\varepsilon/4N$. Call the resulting quantities again γ, N_r and δ_r, since the old ones won't be used again. Choose a new r so $h_r(\phi, \bar{\mathscr{P}}) > h(\phi, \bar{\mathscr{P}}) - \gamma/2$, and a new $\bar{\varepsilon}$ so that if $|\mathscr{P}^* - \bar{\mathscr{P}}| < \bar{\varepsilon}$ then $h_r(\phi, \mathscr{P}^*) > h_r(\phi, \bar{\mathscr{P}}) - \gamma/2$.

Again apply the previous lemma: choose L so $W \subset C_L$ and $L > N_s$ and $\mathscr{Q}_{C_L} \supset \mathscr{P}_1$ with $|\mathscr{P}_1 - \bar{\mathscr{P}}| < \bar{\varepsilon}$. Build $\mathscr{Q}' \subset \bar{\mathscr{P}}_{\mathbf{R}^n}$ so that $(\phi, \bar{\mathscr{P}} \vee \mathscr{Q})$ and $(\phi, \mathscr{P} \vee \mathscr{Q}')$ have a better than δ_r match. Again, the joining can be arranged so that $\bar{\mathscr{P}} \vee \mathscr{Q}_W$ and $\mathscr{P} \vee \mathscr{Q}_W'$ have joint distributions so close that if $\bar{\mathscr{P}}_0' \subset \mathscr{Q}_W'$ corresponds to $\mathscr{P}_0' \subset \mathscr{Q}_W$ then $|\bar{\mathscr{P}}_0' - \bar{\mathscr{P}}_0| < \varepsilon/2$; and also, the partition $\mathscr{Q}_0 \vee \mathscr{Q}'$ in $\mathscr{P}_V \vee \mathscr{Q}'$ and the corresponding $\bar{\mathscr{Q}}_0 \vee \mathscr{Q}$ in $\bar{\mathscr{P}}_V \vee \mathscr{Q}$ have such close joint distributions that $|\mathscr{Q}_0 - \mathscr{Q}'| < |\mathscr{Q}_0 - \mathscr{Q}| + \varepsilon/8 < \varepsilon/4$.

As before, we have some $\bar{\mathscr{Q}} \subset \bar{\mathscr{P}}_{\mathbf{R}^n}$ with $|\bar{\mathscr{Q}} - \mathscr{Q}'| < \varepsilon/2N$ and $(\phi, \bar{\mathscr{Q}}) \approx (\phi, \mathscr{Q})$. Thus the partition $\bar{\bar{\mathscr{P}}}_0 \subset \bar{\mathscr{Q}}_W$ corresponding to $\bar{\mathscr{P}}_0' \subset \mathscr{Q}_W'$ and $\bar{\mathscr{P}}_0 \subset \mathscr{Q}_W$ must satisfy $|\bar{\bar{\mathscr{P}}}_0 - \bar{\mathscr{P}}_0'| < \varepsilon/2$.

Now $\bar{\mathcal{Q}}_{\mathbf{R}^n} \supset \bar{\mathcal{Q}}_W \supset \bar{\bar{\mathcal{P}}}_0$, and $|\bar{\bar{\mathcal{P}}}_0 - \bar{\mathcal{P}}'_0| < \varepsilon/2$, and $|\bar{\mathcal{P}}'_0 - \bar{\mathcal{P}}| < \varepsilon/2$. Thus, $\bar{\mathcal{Q}}_{\mathbf{R}^n} \overset{\varepsilon}{\supset} \bar{\mathcal{P}}$.

Similarly, $|\mathcal{Q} - \bar{\mathcal{Q}}| < |\mathcal{Q} - \mathcal{Q}'_0| + |\mathcal{Q}'_0 - \bar{\mathcal{Q}}_0| + |\bar{\mathcal{Q}}_0 - \mathcal{Q}'| + |\mathcal{Q}' - \bar{\mathcal{Q}}| < \varepsilon/4 + \varepsilon/4 + \varepsilon/4 + \varepsilon/4N < \varepsilon$. □

□

5.12. Corollary. *Any two Bernoulli flows of $\mathbf{R}^n$ of equal finite entropy are isomorphic.*

What is meant by a Bernoulli flow here is that $\psi = \{\phi_v : v \in \mathbf{Z}^n\}$, the integer restriction of ϕ, is a Bernoulli action. This means that ψ may be regarded as the "coordinate shifts" of a family of i.i.d. random variables indexed by $\mathbf{Z}^n$. It is shown in [4] that such ψ are FD, and now Theorem 4.7 shows that ϕ is SFD, and so Theorem 5.9 gives the result.

5.12 is not a new result; it was already proven by D. Lind [5], using an n-dimensional generalization of the argument originally made by Ornstein for $n = 1$. The main point of the present treatment was to introduce other techniques, as a preliminary to their use in new situations.

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